Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for higher forms

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Jolicoeur and Le Guillou have recently proposed [Phys. Rev. A **40**, 5815 (1989)] a very interesting derivation of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy using functional techniques. In this paper we generalized their derivation to higher-form distributions in phase space. At the heart of our method there is a path integral for *classical* mechanics that we put forward some time ago. This path integral describes the dynamics not only of scalar distributions in phase space but also of p-form valued densities [E. Gozzi, M. Reuter, and W. D. Thacker, Phys. Rev. D **40**, 3363 (1989)]. The distribution functions entering this infinite set of coupled integro-differential equations (BBGKY hierarchy) carry a double grading now: besides the level in the hierarchy, they are characterized by their degree as differential forms. The higher forms are related to the dynamics of the Jacobi fields and therefore contain information about the behavior of nearby trajectories. This kind of information could be used in the study of turbulence, for instance.

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In a series of papers [1–5] we have given a path-integral representation of *classical* Hamiltonian dynamics, whose operatorial counterpart generalizes the approach of Koopman and von Neumann [6] to classical mechanics. The measure of that path integral [2] contains a δ function allowing only classical paths to contribute. This measure can be written in the familiar form $\exp(i\tilde{S})$, where the action $\tilde{S}$ not only depends on the (bosonic) phase-space variables $\phi^a(t)$ $a=1,\ldots,2n$, where n is the number of degrees of freedom, but also on new anticommuting ghosts $c^a(t)$, antighosts $\bar{c}_a(t)$, and auxiliary fields $\lambda_a(t)$. It was shown in detail in Refs. [1,2] that the action $\tilde{S}$ has a surprising Becchi-Rouet-Stora (BRS)-like symmetry that "rotates" the phase-space variables ϕ^a into the ghosts c^a. (We assume the reader is familiar with Ref. [2] in the following.) The interesting point is that the ghost-zero modes are precisely the Jacobi fields along the classical trajectories and therefore carry important information about the possible chaotic behavior [7] (exponential instabilities, etc.) of the system. In an operatorial version of the theory [2], c^a and $\bar{c}_a$ become multiplication and derivative operators, respectively, acting on generalized density functions $\tilde{\rho}(\phi^a,c^a,t)$. It is possible to show that the c^a's serve as a basis of the cotangent space $T^*_\phi\mathcal{M}_{2n}$ to the phase space $\mathcal{M}_{2n}$.

The natural way to write the generalized densities $\tilde{\rho}$ is

$$\tilde{\rho}(\phi^a,c^a,t)=\sum_{p=0}^{2n}\frac{1}{p!}\rho^{(p)}_{a_1\cdots a_p}(\phi^a,t)c^{a_1}\cdots c^{a_p}\ . \tag{1}$$

We see that it has the character of an inhomogeneous differential form on $\mathcal{M}_{2n}$ since we may identify c^a with $d\phi^a$. The dynamics of the $\tilde{\rho}$ is given by a "Schrödinger-like" equation:

$$i\partial_t\tilde{\rho}(\phi,c,t)=\tilde{\mathcal{H}}\tilde{\rho}(\phi,c,t)\ . \tag{2}$$

We called it a "Schrödinger-like" equation only because the structure is the same and not because quantum effects may be present: the reader should not forget that everything is *classical* here. This equation is just a generalization of the classical Liouville equation. Here the "super-Hamiltonian" $\tilde{\mathcal{H}}\equiv -il_h$ is essentially the Lie-derivative operator [2]

$$l_h=h^a\partial_a+c^b(\partial_b h^a)(\partial/\partial c^a) \tag{3}$$

for the Hamiltonian vector field h^a of Hamilton's equations,

$$\dot{\phi}^a(t)=h^a(\phi(t))\equiv\omega^{ab}\partial_b H(\phi(t))\ . \tag{4}$$

We use the same notation as in Ref. [2], with ω^{ab} the symplectic tensor that we assume, for simplicity, constant. Thus the antisymmetric tensor fields of Eq. (1) evolve according to

$$\partial_t\rho^{(p)}_{a_1\cdots a_p}(\phi^a,t)=-l_h\rho^{(p)}_{a_1\cdots a_p}(\phi^a,t)\ , \tag{5}$$

where now the Lie derivative l_h acts in the usual way:

$$l_h\rho^{(p)}_{a_1\cdots a_p}=h^b\partial_b\rho^{(p)}_{a_1\cdots a_p}+\partial_{a_1}h^b\rho_{ba_2\cdots a_p}+\partial_{a_2}h^b\rho^{(p)}_{a_1ba_3\cdots a_p}+\cdots\ . \tag{6}$$

In particular, the ordinary densities $\rho(\phi^a)$ are zero forms; from (5) and (6) we see that their time evolution is given by the conventional Liouville equation

$$\partial_t\rho(\phi^a,t)=-\hat{L}\rho(\phi^a,t)\ , \tag{7}$$

where $\hat{L}\equiv h^a\partial_a\equiv l_h|_{p=0}$ is the Liouville operator.

In Refs [1,2] the p-form generalization (5) of the Liouville equation (7) was obtained as the operatorial counterpart of the path integral of *classical* mechanics. In that path integral (for details see Ref. [2]) the integration was over "classical single-particle trajectories" $\phi^a(t)$. In this paper we want to develop a many-particle formalism and it is then natural to replace the functional integration over $\phi^a(t)$ with a "functional integration over fields"

$\tilde{\rho}(\phi^a,c^a,t)$. (This is analogous to what is done when one goes from first to second quantization, but only analogous: in fact, let us not forget that everything here is *classical*). The physics of such a formulation will be completely equivalent to that of Ref. [2] (once an initial many-particle distribution is chosen), but, as a "formal" tool, this formulation of "functional integration over classical fields $\tilde{\rho}$" can be very powerful. This has been pointed out by Jolicoeur and LeGuillou [8] who have used a functional integral representation of Eq. (7) in order to derive the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [9,10] and the Balescu-Lenard collision operator of a high-temperature plasma. It turns out the "*integration over fields*" formulation simplifies these derivations considerably.

The path integral of Ref. [8] is the form

$$Z=\int \mathcal{D}\rho(\phi^a)\mathcal{D}\Lambda(\phi^a,t)\exp\left[i\int d^{2n}\phi\, dt\, \Lambda(\phi^a,t)\times[\partial_t+\hat{L}]\rho(\phi^a,t)\right] . \quad (8)$$

Integrating out the auxiliary field $\Lambda(\phi^a,t)$ we obtain a δ functional $\delta[(\partial_t+\hat{L})\rho]$, so that indeed only the solutions of the Liouville equation (7) can contribute and the physics is not changed by the "integration over fields" procedure with respect to the physics contained in Ref. [2], provided an initial many-particle distribution is chosen for the path integral of Ref. [2]. In this paper we are going to generalize the method of Jolicoeur and Le Guillou by including the higher p-form sectors, i.e., by starting from Eq. (2) instead of Liouville's equation. The measure is chosen such that only classical solutions $\tilde{\rho}_{\rm cl}(\phi^a,c^a,t)$ of Eq. (2) contribute to the path integral on the line of Refs. [1,2],

$$\begin{aligned}\tilde{Z}&=\int \mathcal{D}\tilde{\rho}(\phi^a,c^a,t)\delta[\tilde{\rho}-\tilde{\rho}_{\rm cl}]\\&=\int \mathcal{D}\tilde{\rho}(\phi^a,c^a,t)\det[\partial_t+l_h]\delta[(\partial_t+l_h)] .\end{aligned} \quad (9)$$

Introducing additional superfields $\tilde{\Lambda}(\phi^a,c^a,t)$, $\Psi(\phi^a,c^a,t)$, and $\bar{\Psi}(\phi^a,c^a,t)$ the δ functional and the determinant can be exponentiated as

$$\tilde{Z}=\int \mathcal{D}\tilde{\rho}\mathcal{D}\tilde{\Lambda}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left[i\int d^{2n}\phi\, d^{2n}c\, dt\{\tilde{\Lambda}(\partial_t+l_h)\tilde{\rho}+\bar{\Psi}(\partial_t+l_h)\Psi\}\right] . \quad (10)$$

However, contrary to the "sum-over-classical-phase-space-trajectories" method [1,2], it is not really meaningful to exponentiate the determinant, since (due to the linearity of the field equation $[\partial_t+l_h]\tilde{\rho}=0$) this determinant is independent of $\tilde{\rho}$ and can be absorbed into the overall normalization of $\tilde{Z}$. Therefore in the following we shall work with only the functional integral

$$\tilde{Z}=\int \mathcal{D}\tilde{\rho}(\phi^a,c^a,t)\mathcal{D}\tilde{\Lambda}(\phi^a,c^a,t)\times\exp\left[i\int d^{2n}\phi\, d^{2n}c\, dt\, \tilde{\Lambda}(\phi^a,c^a,t)\times[\partial_t+l_h]\tilde{\rho}(\phi^a,c^a,t)\right] . \quad (11)$$

From its action we can read off the following canonical commutation relation:

$$[\tilde{\rho}(\phi,c,t),\tilde{\Lambda}(\phi',c',t)]=i\delta^{(2n)}(\phi-\phi')\delta^{(2n)}(c-c') . \quad (12)$$

Obviously the auxiliary field $\tilde{\Lambda}$ can be considered as the "momentum" conjugate to $\tilde{\rho}$.

Now we shall use Eq. (11) as the starting point for the derivation of a generalized BBGKY hierarchy which closely follows Ref. [8] but which also includes higher p-form fields. We consider a system consisting of N classical, nonrelativistic, identical particles of mass m. The coordinates on phase space $\mathcal{M}_{2n}$ are written as $\{\phi^a, a=1,2,\ldots,2n=6N\}\equiv\{\hat{\phi}_i, i=1,\ldots,N\}$, where $\hat{\phi}_i\equiv(\mathbf{X}_i,\mathbf{P}_i)$ is the collection of all components referring to one particle. Similarly we write for the ghosts $\{c^a, a=1,2,\ldots,2n=6N\}\equiv\{\hat{c}_i, i=1,\ldots,N\}$, where the position and momentum ghosts of a specific particle are written as $\hat{c}_i\equiv(\xi_i,\pi_i)$. We also shall use greek indices $\mu,\nu,\ldots$ for the one-particle configuration space, i.e., $\mathbf{X}_i=\{X_i^\mu,\mu=1,2,3\}$, etc. Furthermore it is convenient to combine the phase-space coordinates and the ghosts of a given particle into one "supercoordinate": $\chi_i=(\hat{\phi}_i,\hat{c}_i)=(\mathbf{X}_i,\mathbf{P}_i,\xi_i,\pi_i)$, $i=1,\ldots,N$. Accordingly the measure reads $d^{2n}\phi d^{2n}c=d\chi_1\cdots d\chi_N\equiv d^N\chi$. The Hamiltonian is assumed to be of the form

$$H=\sum_{i=1}^{N}\frac{\mathbf{P}_i^2}{2m}+\sum_{\substack{i,j\\i<j}}V(\mathbf{X}_i-\mathbf{X}_j) . \quad (13)$$

The resulting Hamiltonian vector field (Liouville operator) is given by

$$\begin{aligned}\hat{L}&=h^a\partial_a=\omega^{ab}\partial_b H\partial_a\\&=\sum_{i=1}^{N}\left[\frac{P_i^\mu}{m}\frac{\partial}{\partial X_i^\mu}-\sum_{j=1}^{N}{}'\,V_{,\mu}(\mathbf{X}_i-\mathbf{X}_j)\frac{\partial}{\partial P_i^\mu}\right] .\end{aligned} \quad (14)$$

Here $\sum'$ means that we sum between 1 and N but with the term $j=i$ omitted, and $V_{,\mu}\equiv\partial_\mu V$. Similarly the Lie-derivative operator (3) becomes

$$l_h=\sum_{i=1}^{N}l_i^{(1)}+\sum_{i,j}{}'\,l_{ij}^{(2)} \quad (15)$$

with the one-particle operators

$$l_i^{(1)}=(P_i^\mu/m)(\partial/\partial X_i^\mu)+(\pi_i^\mu/m)(\partial/\partial\xi_i^\mu) \quad (16)$$

and the two-particle operators

$$l_{ij}^{(2)}=-\left[V_{,\mu}(\mathbf{X}_i-\mathbf{X}_j)\frac{\partial}{\partial P_i^\mu}+V_{,\mu\nu}(\mathbf{X}_i-\mathbf{X}_j)(\xi_i^\nu-\xi_j^\nu)\frac{\partial}{\partial\pi_i^\mu}\right] . \quad (17)$$

They have the important property that

$$\begin{aligned}&\int d\chi_i l_i^{(1)}F(\chi_1,\ldots,\chi_N)=0 ,\\&\int d\chi_i l_{ij}^{(2)}F(\chi_1,\ldots,\chi_N)=0 \quad (j=1,\ldots,N)\end{aligned} \quad (18)$$

for any function F, since in each term there is one integration which is over a total derivative. In formula (18) there is no sum over i. Because we are considering indistinguishable particles, $\tilde{\rho}$ and $\tilde{\Lambda}$ must be totally symmetric in the "supercoordinates" χ_i. (Note that despite the anticommuting ghosts no antisymmetric, "fermionic," $\tilde{\rho}$ can be allowed since $\tilde{\rho}$ corresponds to $|\psi|^2$ in quantum mechanics.) Being totally symmetric, $\tilde{\Lambda}$ say, can be parametrized in terms of zero-, one-, two-,. . . particle functions $\tilde{\Lambda}_s$ ($s=0,1,\ldots,N$) according to Ref. [9],

$$\tilde{\Lambda}(\chi_1,\chi_2,\ldots,\chi_N,t) = \tilde{\Lambda}_0(t)+\sum_{i=1}^{N}\tilde{\Lambda}_1(\chi_i,t)+\sum_{\substack{i,j\\ i<j}}\tilde{\Lambda}_2(\chi_i,\chi_j,t) + \sum_{\substack{i,j,k\\ i<j<k}}\tilde{\Lambda}_3(\chi_i,\chi_j,\chi_k,t)+\cdots . \quad (19)$$

The functions $\tilde{\Lambda}_s$ are symmetric under permutation of their indices and they cannot be decomposed into a sum of functions depending on less variables. Inserting Eq. (19) into the action $A=\int dt\,\tilde{\mathcal{L}}$ appearing in Eq. (11),

$$\tilde{\mathcal{L}}=\int d^N\chi\tilde{\Lambda}(\chi_1,\ldots,\chi_N,t)[\partial_t+l_h] \times\tilde{\rho}(\chi_1,\ldots,\chi_N,t) \quad (20)$$

we obtain a decomposition of the form

$$\tilde{\mathcal{L}}=\sum_{s=0}^{N}\tilde{\mathcal{L}}_s , \quad (21)$$

where $\tilde{\mathcal{L}}_s$ is the part containing $\tilde{\Lambda}_s$. Changing the integration variables from $\tilde{\Lambda}$ to the set $\{\tilde{\Lambda}_s\}$, and noting that the Jacobian is an irrelevant constant [8], the functional integral becomes

$$\tilde{Z}=\int\mathcal{D}\tilde{\Lambda}_0(t)\mathcal{D}\tilde{\Lambda}_1(\chi_1,t)\mathcal{D}\tilde{\Lambda}_2(\chi_1,\chi_2,t)\cdots \times\exp\left[i\int dt\sum_{s=1}^{N}\tilde{\mathcal{L}}_s\right] . \quad (22)$$

It is convenient to express A in terms of the s-particle reduced distribution functions [9] defined in the usual way:

$$f_s(\chi_1,\ldots,\chi_s)=\frac{N!}{(N-s)!}\int d\chi_{s+1}\cdots d\chi_N \times\tilde{\rho}(\chi_1,\ldots,\chi_N) . \quad (23)$$

What is new is that in our case the $\{f_s\}$ are superfields. The first example is

$$f_1(\chi_1)=N\int d\chi_2\cdots d\chi_N\tilde{\rho}(\chi_1,\chi_2,\ldots,\chi_N) , \quad (24)$$

which has a superfield expansion of the form

$$f_1(\chi_1)=\sum_{p,q=1}^{3} f^{(p,q)}_{(1)i_1\cdots i_p j_1\cdots j_q}(\hat{\phi}_1)\xi_1^{i_1}\cdots\xi_1^{i_p}\pi_1^{j_1}\cdots\pi_1^{j_q} . \quad (25)$$

The path integral (22) gives rise to the (generalized) BBGKY hierarchy in the following way. The Lagrangians $\tilde{\mathcal{L}}_s$ are of the general form

$$\tilde{\mathcal{L}}_s=\int d\chi_1\cdots d\chi_s\tilde{\Lambda}_s(\chi_1,\ldots,\chi_s,t)F_s(\chi_1,\ldots,\chi_s,t) , \quad (26)$$

where F_s is some functional of the f's. Inserting (26) into (22) and performing the $\tilde{\Lambda}_s$ integrations one obtains a product of δ functions which forces all the F_s's to vanish:

$$F_0=0,\quad F_1=0,\quad F_2=0,\ldots . \quad (27)$$

This is a set of coupled integro-differential equations for the f's which coincides with the usual BBGKY hierarchy if one omits the higher ghost sectors [8], and it yields an extension of it in the general case considered here. We list only the first few members of this hierarchy. One finds

$$\tilde{\mathcal{L}}_0=\tilde{\Lambda}_0(t)\frac{d}{dt}\int d^{2n}\phi\,\rho(\phi,t) , \quad (28)$$

$$\tilde{\mathcal{L}}_1=\int d\chi_1\tilde{\Lambda}_1(\chi_1,t)\left[(\partial_t+l_1^{(1)})f_1(\chi_1,t) + \int d\chi_2 l_{12}^{(2)}f_2(\chi_1,\chi_2,t)\right] , \quad (29)$$

$$\tilde{\mathcal{L}}_2=\tfrac{1}{2}\int d\chi_1 d\chi_2\tilde{\Lambda}_2(\chi_1,\chi_2,t) \times[(\partial_t+l_1^{(1)}+l_2^{(1)}+l_{12}^{(2)}+l_{21}^{(2)})f_2(\chi_1,\chi_2,t) +(l_{13}^{(2)}+l_{23}^{(2)})f_3(\chi_1,\chi_2,\chi_3,t)] . \quad (30)$$

In deriving these equation we made use of Eqs. (15), (18), and (23), and of the symmetry of $\tilde{\Lambda}_s$ and $\tilde{\rho}$. In Eq. (28) the ordinary scalar density $\rho(\phi,t)$ is defined by $\tilde{\rho}=\rho(\phi)c^1c^2\cdots c^{2n}+\cdots$ where the $\cdots$ denote terms with less than $2n$ ghosts. Thus the lowest equation of the hierarchy, $F_0=0$, expresses the conservation of the total probability:

$$\frac{d}{dt}\int d^{2n}\phi\,\rho(\phi,t)=0 . \quad (31)$$

The equations $F_1=0$ and $F_2=0$ read

$$[\partial_t+(P_l^\mu/m)(\partial/\partial X_l^\mu)+(\pi_1^\mu/m)(\partial/\partial\xi_1^\mu)]f_1(\chi_1,t) -\int d\chi_2[V_{,\mu}(\mathbf{X}_1-\mathbf{X}_2)(\partial/\partial P_1^\mu)+V_{,\mu\nu}(\mathbf{X}_1-\mathbf{X}_2)(\xi_1^\nu-\xi_2^\nu)(\partial/\partial\pi_1^\mu)]f_2(\chi_1,\chi_2,t)=0 , \quad (32)$$

$$\begin{aligned}&[\partial_t+(P_1^\mu/m)(\partial/\partial X_1^\mu)+(\pi_1^\mu/m)(\partial/\partial\xi_1^\mu)+(P_2^\mu/m)(\partial/\partial X_2^\mu)+(\pi_2^\mu/m)(\partial/\partial\xi_2^\mu)-V_{,\mu}(\mathbf{X}_1-\mathbf{X}_2)(\partial/\partial P_1^\mu)\\ &-V_{,\mu\nu}(\mathbf{X}_1-\mathbf{X}_2)(\xi_1^\nu-\xi_2^\nu)(\partial/\partial\pi_1^\mu)-V_{,\mu}(\mathbf{X}_2-\mathbf{X}_1)(\partial/\partial P_2^\mu)-V_{,\mu\nu}(\mathbf{X}_2-\mathbf{X}_1)(\xi_2^\nu-\xi_1^\nu)(\partial/\partial\pi_2^\mu)]f_2(\chi_1,\chi_2,t)\\ &-\int d\chi_3[V_{,\mu}(\mathbf{X}_1-\mathbf{X}_3)(\partial/\partial P_1^\mu)+V_{,\mu\nu}(\mathbf{X}_1-\mathbf{X}_3)(\xi_1^\nu-\xi_3^\nu)(\partial/\partial\pi_1^\mu)+V_{,\mu}(\mathbf{X}_2-\mathbf{X}_3)(\partial/\partial P_2^\mu)\\ &+V_{,\mu\nu}(\mathbf{X}_1-\mathbf{X}_3)(\xi_2^\nu-\xi_3^\nu)(\partial/\partial\pi_2^\mu)]f_3(\chi_1,\chi_2,\chi_3,t)=0 .\end{aligned} \quad (33)$$

We can continue this way for F_3, F_4, etc. These equations could be written as conventional integro-differential equations, i.e., as equations without Grassmann variables, by inserting the superfield expansions [like Eq. (25), for instance] for the distribution functions f_s. The result would be a set of equations for the expansion coefficients $\{f^{(p,q)}_{(1)i_1\cdots i_p j_1\cdots j_q}(\hat{\phi},t),\cdots\}$. These functions are characterized by the level s in the hierarchy ($s=0,\ldots,N$), and by their degree as tensor fields, i.e., their ghost numbers with respect to ξ_i^μ and π_i^μ.

The reader might ask which is the physical meaning of the higher p-form fields. Let us recall the role played by the ghost-zero modes $c^a(t)$ in Ref. [2]. They are solutions of the classical Jacobi equation and thus can be written as

$$c^a(t)=S^a_b(t,\phi_0)c^b(0)\ , \tag{34}$$

where the symplectic matrix S^a_b is a solution of

$$[\partial_t\delta^a_b-\partial_b h^a(\phi_{\rm cl}(t,\phi_0))]S^b_c(t,\phi_0)=0 \tag{35}$$

and $\phi_{\rm cl}(t,\phi_0)$ is the solution of Hamilton's equation with $\phi_{\rm cl}(0,\phi_0)=\phi_0$. Knowing the Jacobi field $c^a(t)$ around some classical solution $\phi_{\rm cl}$ we can use the associated matrix S^a_b to construct a singular solution of Eq. (5), for $p=1$, say

$$\rho^{(1)}_a(\phi,t)=S^{-1}(t,\phi_0)^b_a\rho_b(0)\delta^{2n}(\phi-\phi_{\rm cl}(t,\phi_0))\ . \tag{36}$$

This field has support only along the trajectory $\phi_{\rm cl}(t,\phi_0)$. It describes the evolution of a nearby classical trajectory with initial separation $\rho_b(0)$ from $\phi_{\rm cl}(0,\phi_0)$. Nonsingular solution of (5) can be obtained from (36) by smoothing out the initial conditions ϕ_0 and $\rho_b(0)$ with some arbitrary distribution $P(\phi_0,\rho_b(0))$. Then the resulting 1-form field $\rho^{(1)}_a(\phi,t)$ encodes information about the Jacobi fields around all possible classical trajectories, i.e., trajectories starting at any point of phase space. For a system of identical particles we can introduce the reduced distribution functions f_s as in the conventional $p=0$ case. Then a function like $f^{(1,0)}_{(1)i}(\hat{\phi}_1)$, say, which appears in the sum of Eq. (25), describes the Jacobi fields (in the ξ^μ direction) on the one-particle phase space $\{\mathbf{X}_1,\mathbf{P}_1\}$. It tells us how an initial shift of the $\mathbf{X}_1$ coordinates is propagated in time. Similarly $f^{(0,1)}_{(1)j}(\hat{\phi}_1)$ describes the effect of an initial momentum shift, etc. This kind of information might be important for the study of turbulence or the chaotic behavior of a plasma, for example. Work is in progress to "translate" all the important concepts related to chaos in terms of our formalism. In particular we have found a way to rewrite the highest Liaupunov exponents in terms of ghost condensate. This work [11] is in preparation.

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APPENDIX

In this appendix we will show in detail how to derive Eq. (29). Let us start from the general definition of $\mathcal{L}_1$,

$$\mathcal{L}_1=\sum_{i=1}^N\int d\chi_1\cdots d\chi_N\Lambda_1(\chi_i,t)[\partial_t+l_h]\times\tilde{\rho}(\chi_1,\chi_2,\ldots,\chi_N,t)\ . \tag{A1}$$

Due to the fact that $[\partial_t+l_h]\tilde{\rho}(\chi_1,\chi_2,\ldots,\chi_N,t)$ is symmetric in all χ_i, the formula above reduces to

$$\mathcal{L}_1=N\int d\chi_1,\Lambda(\chi_1 t)\int d\chi_2\cdots d\chi_N\times\left[\partial_t+\sum_{i=1}^N l_i^{(1)}+\sum_{i,j}{}' l_{ij}^{(2)}\right]\tilde{\rho}(\chi_1,\ldots\chi_N,t)\ . \tag{A2}$$

Because of formula (18) only the term $i=1$ contributes to the formula above, so we are left with

$$\mathcal{L}_1=\int d\chi_1\Lambda_1(\chi_1,t)\left[\{\partial_t+l^{(1)}\}N\int d\chi_2\cdots d\chi_N\tilde{\rho}(\chi_1,\ldots,\chi_N,t)+\sum_{j=2}^N N\int d\chi_2,\ldots,d\chi_N l^{(2)}_{1j}\tilde{\rho}(\chi_1,\ldots,\chi_N,t)\right]\ . \tag{A3}$$

If we note that $\sum_{j=2}^N\int d\chi_2,\ldots,d\chi_N l^{(2)}_{1j}\tilde{\rho}(\chi_1,\ldots,\chi_N,t)$ yields $(N-1)$ equal terms, and so it is equal to $\int d\chi_2 l^{(2)}_{12}N(N-1)\int d\chi_3\cdots d\chi_N\tilde{\rho}(\chi_1,\ldots,\chi_N,t)$, and if we call $N\int d\chi_2,\ldots,d\chi_N\tilde{\rho}(\chi_1,\ldots,\chi_N,t)\equiv f_1(\chi_1,t)$ and $N(N-1)\int d\chi_3\cdots d\chi_N\tilde{\rho}(\chi_1,\ldots,\chi_N,t)\equiv f_2(\chi_1,\chi_2,t)$, then Eq. (A3) can be written as

$$\mathcal{L}_1=\int d\chi_1\Lambda_1(\chi_1,t)\left[\{\partial_t+l_1^{(1)}\}f_1(\chi_1,t)+\int d\chi_2 l^{(2)}_{12}f_2(\chi_1,\chi_2,t)\right]\ .$$

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